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On the Quantum Hamiltonian for a Charge in an Electromagnetic Field, in Generalized Coordinates

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When one considers the quantum Hamiltonian for a charge in an electromagnetic field in *Cartesian coordinates*, no problem arises concerning the ordering of the quantum operators. However, if one considers the quantum Hamiltonian operator for a charge in an electromagnetic field in *generalized coordinates*, the question on the ordering problem becomes very apparent.

One can easily find the *classical* Hamiltonian, H_c, for the charge in the electromagnetic field in generalized coordinates $\{q_i\}$ $(x_i = x_i(q_1, q_2, \ldots))$ to be given as‡

$$H_c = \frac{1}{2m} g^{mn} p_m p_n + \frac{e}{2mc} A_i(\hat{e}_i \cdot \hat{x}_t) A_j(\hat{e}_j \cdot \hat{x}_t) - \frac{e}{cm} \frac{\partial q_m}{\partial x_r} (\hat{e}_i \cdot \hat{x}_r) A_i p_m + e\phi \quad (1)$$

where

$$g^{mn} = \frac{\partial q^m}{\partial x_t} \frac{\partial q^n}{\partial x_t}.$$

A_i is the vector potential component along the $\hat{e}_i$ (unit-vector) axis.
$\hat{x}_t$ is the unit vector along the respective Cartesian axis.
p_i is the canonical generalized momentum.
e is the charge on the electron, m is the mass, c is the speed of light.
ϕ is the electromagnetic scalar potential.

In making the transition to quantum theory, one is faced with the problem of what to use for the quantum operator for the first and third term of equation (1). With $p_i = -i\hbar \partial/\partial q_i$, it is found that (Gruber, 1972a) the first term in equation (1), that is,

$$\frac{1}{2m} g^{mn} p_m p_n$$

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‡ Using Einstein summation convention.

quantum mechanically becomes

$$\frac{1}{2m} p_m\dagger\, g^{mn} p_n$$

where $p_m\dagger$ denotes the adjoint (Gruber, 1971, 1972a) of p_m.

The third term of equation (1), which is

$$-\frac{e}{cm}\frac{\partial q_m}{\partial x_r}(\hat{e}_i \cdot \hat{x}_r) A_i p_m$$

quantum mechanically becomes (Gruber, 1972b)

$$\left[-\frac{e}{cm}\frac{\partial q_m}{\partial x_r}(\hat{e}_i \cdot \hat{x}_r) A_i p_m\right]^H \qquad (p_m = -i\hbar\, \partial/\partial q_m)$$

the Hermitian part of the operator

$$-\frac{e}{cm}\frac{\partial q_m}{\partial x_r}(\hat{e}_i \cdot \hat{x}_r) A_i p_m$$

All the other terms in equation (1) quantum mechanically remain of course as their respective functions. Thus the quantum mechanical operator for the Hamiltonian of a charge in an electromagnetic field, in generalized coordinates, H, is given as

$$H = \frac{1}{2m} p_m\dagger\, g^{mn} p_n + \frac{e}{2mc} A_i(\hat{e}_i \cdot \hat{x}_t) A_j(\hat{e}_j \cdot \hat{x}_t) - \left[\frac{e}{cm}\frac{\partial q_m}{\partial x_r}(\hat{e}_i \cdot \hat{x}_r) A_i p_m\right]^H + e\phi$$

Note that since p_i is not self-adjoint, the third term in equation (1) cannot simply be symmetrized to give the correct quantum mechanical correspondence.

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